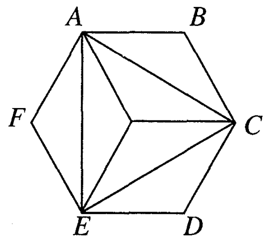
UK Junior Mathematical Olympiad 2000 Solutions

A1 $2000 + 1999 \times 2000 = (1 + 1999) \times 2000 = 2000 \times 2000 = 4\,000\,000$.

4000000

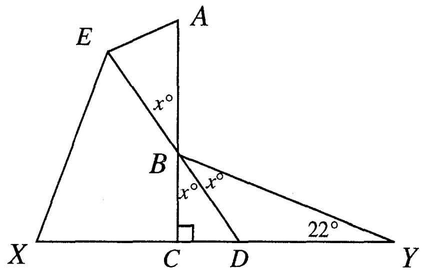
A2 **$\frac{2}{3}$** The regular hexagon may be divided up into six congruent triangles, as shown. The kite $ABCE$ is made up of four of these.



A3 **7 mm** The area of the square = $0.5\text{ cm}^2 = 50\text{ mm}^2$.
Therefore the side of the square = $\sqrt{50}$ mm = 7 mm (to the nearest mm).

A4 **£1.20** 20% of 30% of 40% of £50 = $\frac{2}{10} \times \frac{3}{10} \times \frac{4}{10} \times £50 = \frac{24}{1000} \times £50 = £\frac{1200}{1000} = £1.20$.

A5 **34** Produce AB so that it meets XY at C. Then, since AB is vertical and XY horizontal, $\angle ACY$ is a right angle. Now $\angle CBD = \angle EBA = x°$ (vertically opposite angles).
Therefore, in ΔCBY :
$22 + x + x + 90 = 180$
i.e. $2x = 68$ i.e. $x = 34$.



A6 **5 p.m.** The Mad Hatter's watch next showed the correct time when the minute hand had gained exactly one hour. This took two hours to do since the minute hand makes three complete revolutions every two hours. Hence it next showed the correct time at 5 p.m.

A7 **60** The number of different ways in which the medals may be allocated is clearly the same as it would be if Mr. Turner first chose one pupil for the Bronze medal and then two pupils for the Silver medals. The three remaining pupils would then receive the Gold medals.

There are six possible ways to allocate the Bronze medal. For each choice of the Bronze medallist, the two Silver medallists have to be chosen from the 5 remaining pupils: here there are 5 ways to choose the first Silver medallist A, and for each such choice there are four ways to choose the second Silver medallist B. So there are 5×4 ways of choosing the Silver medallists "in order". However, AB and BA are then counted separately, so, once the Bronze medallist has been chosen, there are $\frac{5\times4}{2} = 10$ unordered ways to choose the two Silver medallists. The Gold medallists are then determined. Hence there are 6×10 ways to allocate the medals to the six pupils.

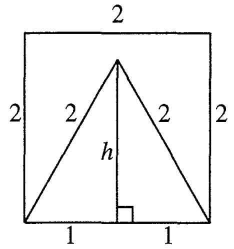
A8 **$(4 - \sqrt{3})$ cm^2**

Let the height of the equilateral triangle be h cm.

Then, by the Theorem of Pythagoras: $h^2 + 1^2 = 2^2$ i.e. $h = \sqrt{3}$.

Hence area of triangle $= \frac{1}{2} \times 2 \times \sqrt{3}\,\text{cm}^2 = \sqrt{3}\,\text{cm}^2$.

Therefore, the area of the square which remains is $(4 - \sqrt{3})$ cm^2.



A9 **5600**

On Tuesday, the machine printed 700 more colour photographs than on Monday, but 1050 fewer monochrome photographs. In the time it took to print 2800 colour photographs, therefore, the number of monochrome photographs it could have printed is $4 \times 1050 = 4200$. Hence the number of monochrome photographs the machine printed on Wednesday is $4200 + 1400 = 5600$.

A10 **9 hours**

One gardener takes four hours to dig one flower bed of diameter four metres. Now the time taken to dig a circular flower bed is directly proportional to its area and hence directly proportional to the square of its diameter.

Thus: $\dfrac{\text{time taken to dig bed of diameter six metres}}{\text{time taken to dig bed of diameter four metres}} = \left(\dfrac{6}{4}\right)^2 = \left(\dfrac{3}{2}\right)^2 = \dfrac{9}{4}$.

Hence the time taken for one gardener to dig a bed of diameter six metres is $\frac{9}{4} \times 4$ hours $= 9$ hours.

The time taken for six gardeners to dig six flower beds is the same as the time it takes one gardener to dig one bed of the same size and hence the answer is 9 hours.

Section B

B1 (i) Let the number of blocks on the top step be t and the length of each step be l. Then:

Number of blocks in a six-step 'staircase'

$= t + (t + l) + (t + 2l) + (t + 3l) + (t + 4l) + (t + 5l) = 6t + 15l$.

Hence $6t + 15l = 90$ i.e. $2t + 5l = 30$.

We deduce that l must be even (*why?*) and also that $l < 6$ (since $t \leqslant 0$ when $l \geqslant 6$).

When $l = 2$: $2t + 10 = 30$ i.e. $t = 10$.

When $l = 4$: $2t + 20 = 30$ i.e. $t = 5$.

Thus there are exactly two different ways of building six-step 'staircases' using all the blocks: 10, 12, 14, 16, 18, 20 and 5, 9, 13, 17, 21, 25.

(ii) The number of blocks in a seven-step 'staircase' $= 6t + 15l + (t + 6l) = 7t + 21l = 7(t + 3l)$.

Thus we would require $7(t + 3l) = 90$. However, 90 is not a multiple of 7 and therefore a seven-step 'staircase' using all 90 blocks is not possible.

B2 A sensible place to start would be to try to find 1 Down since there are only five three-digit cubes, namely 125, 216, 343, 512 and 729.

We may rule out 343 and 512 because 4 Across is a square and so cannot end in either '3' or '2'. Also, 729 may be ruled out since 1 Across cannot begin with '7'.

If 1 Down is 216, then 1 Across must be 25. However, 2 Down cannot start with '5' and hence 216 can also be eliminated.

Any solution, therefore, must have 125 as 1 Down. This means that 1 Across and 2 Down must be 16 and 64 respectively: there are no other correct possibilities. The only two-digit square ending in '5' is 25 and hence 4 across must be 25.

We note that 3 Across is a three-digit square ending in 24 and the only possibility is $18^2 = 324$.

We have now filled in all the squares, with there being no other correct possibilities, but we must check that 3 Down is correct. As $32 = 8 \times 4 = 2^3 \times 2^2$, it is indeed a cube times a square and therefore we deduce that there is exactly one solution to the crossnumber.

B3 ΔDFC is isosceles ($CF = CD$).

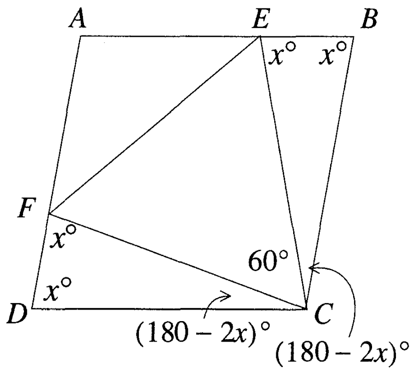
Hence $\angle DFC = \angle FDC = x°$.

Hence $\angle FCD = (180 - 2x)°$ (angle sum of triangle).

Now $\angle EBC = \angle FDC = x°$ (opposite angles of a parallelogram) and ΔEBC is isosceles ($CE = CB$).

Hence $\angle BEC = x°$ and $\angle ECB = (180 - 2x)°$.

Lines AD and BC are parallel and hence $\angle ADC + \angle BCD = 180°$.



Therefore: $x + 2(180 - 2x) + 60 = 180$ i.e. $420 - 3x = 180$ i.e. $3x = 240$ so $x = 80$.

B4 Each of the two three-digits numbers is less than 1000. Their sum, therefore, is less than 2000 and we can deduce that $S = 1$. We note now that C + O must equal either 1 or 11. However, since neither C nor O can have the same value as S (i.e. 1), their sum cannot be 1 and hence C + O = 11. There is a carry of 1 into the tens column, therefore, and, since 1 + M + M cannot equal M, we may deduce that 1 + M + M = M + 10 i.e. M = 9.

This means that there is a carry of 1 into the hundreds column and hence 1 + J + J = U + 10.

We must, therefore, find all possible values of C, O, J and U which satisfy C + O = 11 and U = 2J – 9, remembering that none of these four letters may be either 1 or 9.

From the equation U = 2J – 9, we deduce that J ⩾ 5.

If J = 5 then U = 1 (impossible).

If J = 6 then U = 3. Hence C = 4 and O = 7 or vice versa.

If J = 7 then U = 5. Hence C = 3 and O = 8 or vice versa.

If J = 8 then U = 7. Hence C = 5 and O = 6 or vice versa.

Thus there are six different solutions:

694	697	793	798	895	896
697	694	798	793	896	895
1391	1391	1591	1591	1791	1791

B5 (i) Let the smallest of the three integers be s. Then the others are $(s + 1)$ and $(s + 2)$.

Their sum $= s + (s + 1) + (s + 2) = 3s + 3 = 3(s + 1)$ is therefore always divisible by 3.

However, a method which will be more useful for part (iii) of this question is to let the middle number be m. Then the smallest of the three numbers is $(m - 1)$ and the largest $(m + 1)$.

Their sum $= (m - 1) + m + (m + 1) = 3m$ is therefore always divisible by 3.

(ii) No. It is not true. In part (i) where we are required to explain why something is always true, we need to use algebra. However, when we need to show that a statement is not always true, one example (known as a 'counter-example') is sufficient.

Note that $1 + 2 + 3 + 4 = 10$, which is not divisible by 4, and thus we have shown that the sum of four consecutive positive integers is not always divisible by 4.

Note:

We could, if required, show that the sum of four consecutive integers is **never** divisible by 4. Let the smallest of the four integers be s. Then the others are $(s + 1)$, $(s + 2)$ and $(s + 3)$. Their sum $= s + (s + 1) + (s + 2) + (s + 3)$ $= 4s + 6$, which is not divisible by 4.

(iii) A little investigation appears to show that the sum of k integers is divisible by k when k is odd, but not when k is even. However, as in part (i), we must use algebra to prove that this is **always** the case.

Consider the case when k is odd: let $k = 2n + 1$ where n is a positive integer.

Let the middle number of these k numbers be m. Then the smallest of the numbers is $(m - n)$ and the largest $(m + n)$. Let their sum be S. Then:

$$S = (m - n) + (m - (n - 1)) + (m - (n - 2)) + \ldots + (m - 1) + m + (m + 1) + \ldots + (m + (n - 2)) + (m + (n - 1)) + (m + n).$$

Note that $(m - n) + (m + n) = 2m$; $(m - (n - 1)) + (m + (n - 1)) = 2m$; $(m - (n - 2)) + (m + (n - 2)) = 2m$ etc.

There are n pairs of terms which each sum to $2m$ and one remaining term, the middle term, m. Hence: $S = n \times 2m + m = (2n + 1)m = km$.

We deduce that the sum of k consecutive integers is always divisible by k when k is odd.

Consider the case when k is even. Our conjecture is that the sum of k consecutive integers is **not** always divisible by k when k is even. To prove that this is correct we need only to find one counter-example for each value of k.

Let $k = 2n$ where n is a positive integer.

$$\begin{array}{lcccccccccc} \text{Let} & S = & 1 & + & 2 & + & 3 & + \ldots + & (2n-1) & + & 2n \\ \text{Then} & S = & 2n & + & (2n-1) & + & (2n-2) & + \ldots + & 2 & + & 1 \end{array}$$

Adding these two equations gives:

$$2S = (2n + 1) + (2n + 1) + (2n + 1) + \ldots + (2n + 1) + (2n + 1)$$

$$= 2n(2n + 1).$$

Therefore $S = n(2n + 1)$, which is an **odd** multiple of n and therefore cannot be a multiple of $2n$. Thus when k is even, the sum of the first k positive integers is not divisible by k and so we deduce that the sum of k consecutive positive integers is not always divisible by k when k is even.

(It is possible to prove that the sum of k consecutive integers is never divisible by k when k is even, but this is not necessary in this case.)

Thus the sum of k consecutive integers is divisible by k if and only if k is odd.

B6 (i) X chooses 2 first. Whatever Y chooses, X can now make the total 5 i.e. if Y chooses 1 then X chooses 2 and vice versa. X repeats this tactic, choosing the number which Y does not choose, thereby ensuring that after four more turns each the totals are 8, 11, 14 and 17 respectively. If Y now chooses 1, making the total 18, X chooses 2 to make the total 20 and win the game. Alternatively, if Y chooses 2, then X chooses 1 and still wins.

(ii) A similar tactic is required for the revised game. The important totals, i.e. the 'winning positions', this time are 3, 6, 9, 12 and 15; a player achieving any of these totals can win the game by best play. However, a player who makes the total 1 or 2 or 4 or 5 or 7 etc. gives the opponent the opportunity of establishing and maintaining a 'winning position'. Since Y can achieve either 3 or 6 after one turn but X cannot, it follows that it is Y who can always win. This is shown in the following table.

		A			B			C	
Original Total	0	0	0	3	3	3	6	6	6
X chooses	1	2	4	1	2	4	1	2	4
Y chooses	2	4	2	2	4	2	2	4	2
New Total	3	6	6	6	9	9	9	12	12
Comment	Go to B	Go to C	Go to C	Go to C	Go to D	Go to D	Go to D	Go to E	Go to E

		D			E			F	
Original Total	9	9	9	12	12	12	15	15	15
X chooses	1	2	4	1	2	4	1	2	4
Y chooses	2	4	2	2	1	4	4	4	1
New Total	12	15	15	15	15	20	20	21	20
Comment	Go to E	Go to F	Go to F	Go to F	Go to F	Y wins	Y wins	Y wins	Y wins

(Other choices are available to Y in some cases e.g. column 2 of B could be:

3	2	1	6	Go to C)

Note that game (i) is equivalent to the game in which players take turns to take either one or two matches from an original pile of 20 matches with the winner being the player who takes the last match.